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THE TRANSFER OF HEAT ACROSS A TURBULENT BOUNDARY LAYER AT VERY HIGH PRANDTL NUMBERS

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Abstract—The paper considers a problem which was first treated mathematically by Lighthill in a different physical context. Solutions are provided for the limiting case of forced convection across a turbulent boundary layer when $Pr \to \infty$, i.e. when the thermal boundary layer is wholly confined within the laminar sublayer whose velocity profile is linear.

The case of a flat plate with a uniform temperature or with one step in temperature is treated in great detail, and a convenient tabulation of formulae for a number of cases is provided.

The case of a variable wall temperature is solved in two ways. First, the temperature distribution is replaced by a sequence of steps and superposition is used. Secondly, an exact analytic solution is given for the case when the temperature function consists of a step followed by a distribution given analytically. In the latter case, closed-form equations are given for a polynomial temperature variation of which a linear temperature variation is a special case.

LIST OF SYMBOLS

A,	constant;
A_n,	constant;
a,	thermal diffusivity;
B,	constant;
b,	width of heated portion;
C_1,	constant;
C_2,	constant;
c_f,	coefficient of skin friction;
k,	thermal conductivity;
L,	length of laminar portion of boundary layer;
l,	length of plate;
Nu_l,	average Nusselt number; based on total length l;
Nu_x,	local Nusselt number; based on current length x;
Nu'_x,	local Nusselt number; based on distance $x - x_0$, equation (22);
n,	integer exponent in power series for temperature distribution;
Pr,	Prandtl number;
$\dot{Q}_w$,	rate of heat flow per unit width and time measured at wall;
$\dot{q}_w$,	heat flux at wall;
Re_{crit},	Reynolds number based on length L at transition;
Re_l,	Reynolds number; based on total length l;
Re_x,	local Reynolds number; based on current length x;
Re'_x,	local Reynolds number; based on $x - x_0$, equation (22);
St_x,	local Stanton number; based on current length x;
T,	temperature;
T_∞,	free-stream temperature;
T_w,	temperature at wall;
$\bar{U}_\infty$,	free-stream turbulent velocity;
u,	average longitudinal velocity component in boundary layer;
v,	average transverse velocity component in boundary layer;
v_*,	friction velocity;
x,	longitudinal co-ordinate;
x_0,	co-ordinate at temperature step;
x^+,	stretched longitudinal co-ordinate, equation (11);
x_b^+,	stretched width of heated portion;
$x_{b,n}^+$,	stretched width of step in temperature, equation (43);
y,	transverse co-ordinate;

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y^+, dimensionless transverse co-ordinate, equation (8).

Greek symbols

- α, Blasius constant, equation (24); also parameter in incomplete gamma function, equation (17a);
- $\Gamma(\alpha)$, gamma function of argument α;
- $\Gamma(\alpha, \eta)$, incomplete, complementary gamma function of argument η with parameter α, equation (36);
- $\gamma(\alpha, \eta)$, incomplete gamma function of argument η with parameter α, equation (17a);
- $\delta(x)$, velocity boundary layer thickness;
- $\delta_T(x)$, thermal boundary layer thickness;
- η, similarity parameter, equation (13);
- Θ, temperature ratio, equation (33);
- θ, temperature difference, equation (2);
- θ_∞, difference between wall- and free-stream temperature;
- ϑ, temperature difference, equation (32);
- ϑ_w, difference between wall- and free-stream temperature;
- $\vartheta_{w,n}$, temperature difference for n-th step;
- $\Delta\vartheta_n$, temperature step in step-wise approximation;
- λ, dummy variable of integration;
- ν, kinematic viscosity;
- ξ, dummy variable of integration;
- ρ, density;
- σ, variable defined in equation (10);
- τ, shearing stress;
- τ_w, shearing stress at wall;
- ψ, stream function.

1. INTRODUCTION

THERE exists one limiting case of forced convection across a turbulent boundary layer which can be solved analytically entirely from first principles. It is the case when the thermal boundary layer is confined entirely within the laminar sublayer. Such conditions prevail very close to the beginning of a thermal entry length in a boundary layer at all Prandtl numbers, or over the whole of the downstream length of a thermal boundary layer at very high Prandtl numbers, or more precisely when $Pr \to \infty$.

The attendant mathematical problem was first solved by Lighthill [1] in an entirely different physical context, namely in connection with the calculation of heat transfer rates across laminar, compressible boundary layers. The relevance of Lighthill's solution to turbulent boundary layers was first pointed out by Spalding [2].

In the present paper we shall give an elementary solution of the problem and work out its implications to a point when they can be applied.

The amenability of the problem to such a treatment turns on two circumstances. First, there is no need to consider the whole of the velocity boundary layer, and attention can be confined to the laminar sublayer only, because the boundary condition for temperature "at infinity" is attained very fast, owing to the boundary layer nature of the solution for the temperature profile. This permits us to base the analysis on the laminar form of the energy equation which can now be written

$$u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y} = a\frac{\partial^2\theta}{\partial y^2}, \tag{1}$$

using the notation of Fig. 1.

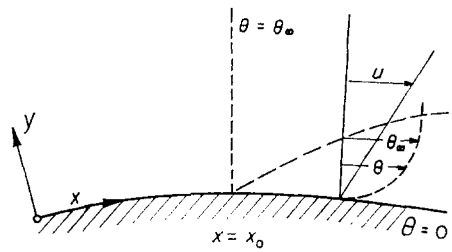


FIG. 1. System of co-ordinates and notation.

Here

$$\theta = T_w - T \tag{2}$$

is the temperature difference between that at a point x,y and at the wall. The boundary conditions for θ are

$$\left.\begin{aligned} \theta &= 0 \text{ at } y = 0 \text{ and all } x \geq x_0 \\ \theta &= \theta_\infty \text{ at } x = x_0 \text{ and all } y \geq 0 \\ \theta &= \theta_\infty \text{ at } y = \infty \text{ and all } x \geq x_0 \end{aligned}\right\} \tag{1a}$$

with

$$\theta_\infty = T_w - T_\infty \tag{1b}$$

denoting the overall temperature difference, not necessarily constant, across the boundary layer. The second simplification consists in the fact that the variation of u with y is linear, so that the stream function can be written

$$\psi = \frac{1}{2}\frac{v_*^2}{\nu}y^2$$

where

$$v_*(x) = \{\tau_w(x)/\rho\}^{1/2} = \bar{U}_\infty(\tfrac{1}{2}c_f)^{1/2} \quad (3)$$

is the friction velocity,

$$\tau_w = \tfrac{1}{2}\rho\bar{U}_\infty^2 c_f, \quad (4)$$

denoting the skin friction at the wall. Hence

$$u = \frac{v_*^2}{\nu}y \quad \text{and} \quad v = -\frac{1}{2}\frac{\mathrm{d}(v_*^2)/\mathrm{d}x}{\nu}y^2. \quad (5)$$

Thus the equation to be solved is

$$\frac{v_*^2}{\nu}y\frac{\partial\theta}{\partial x} - \frac{1}{2}\frac{\mathrm{d}(v_*^2)/\mathrm{d}x}{\nu}y^2\frac{\partial\theta}{\partial y} = a\frac{\partial^2\theta}{\partial y^2} \quad (6)$$

subject to the boundary conditions (1a).

The preceding equations have been written on the assumption that the physical properties of the fluid, its density ρ, kinematic viscosity ν, and thermal diffusivity a are independent of temperature. In general, this is a poor assumption for liquids, because when their Prandtl number is large, their properties vary strongly with temperature. However, the variation of such properties with temperature is complex and cannot be taken into account at present. In applications it is therefore necessary to choose proper mean values.

In all problems of practical interest, the free-stream temperature T_∞ can be assumed constant, but the wall temperature T_w need not be constant. Thus θ_∞ may be variable, and a prescribed function of x. We shall consider both cases, and in both cases we shall provide solutions which are valid from point $x = x_0$ at which the thermal boundary layer is assumed to begin its development. Since equation (6) is linear, superposition can be employed to develop more complex solutions from simple ones.

2. STEP IN WALL TEMPERATURE VARIATION

We begin by considering the case when the wall temperature T_w is constant being equal to T_∞ for $x < x_0$ and to another value T_w for $x \geq x_0$, undergoing a jump at $x = x_0$. Hence θ_∞ is now constant.

From the nature of the problem it can be foreseen that the solution must appear in the form of a family of self-similar temperature profiles, because there exists no characteristic length which governs the development of the thermal boundary layer. Hence, the problem must be reducible to an ordinary differential equation, it must be expressible in terms of a single variable, say η, and the only difficulty lies in indicating the required transformation.

By introducing the ratio

$$\Theta = \frac{T_w - T}{T_w - T_\infty} = \frac{\theta}{\theta_\infty} \quad (7)$$

we can first reduce the values of the dependent variable in the boundary conditions to pure numbers, namely

$$\left.\begin{aligned} &\Theta = 0 \text{ at } y = 0 \text{ and all } x \geq x_0 \\ &\Theta = 1 \text{ at } x = x_0 \text{ and all } y \geq 0 \\ &\Theta = 1 \text{ at } y = \infty \text{ and all } x \geq x_0 \end{aligned}\right\} \quad (7a)$$

and the partial differential equation is still

$$\frac{v_*^2}{\nu}y\frac{\partial\Theta}{\partial x} - \frac{1}{2}\frac{\mathrm{d}(v_*^2)/\mathrm{d}x}{\nu}y^2\frac{\partial\Theta}{\partial y} = a\frac{\partial^2\Theta}{\partial y^2}. \quad (7b)$$

It is recalled that the energy equation can be simplified considerably by the application of the von Mises transformation [3, 4] in which the dependent variable is expressed in terms of x, and the stream function ψ. If this form were written it would become immediately apparent that in the present case it is more convenient to use the square root of the stream function, $\psi^{1/2}$, as the second independent variable, making it dimensionless by the factor $\nu^{-1/2}$. In view of equation (3) this suggests the choice of

$$y^+ = \frac{yv_*}{\nu} \quad (8)$$

as the independent variable. Substitution into (7b) leads to

$$y^+\frac{\nu}{v_*}\frac{\partial\Theta}{\partial x} = \frac{a}{\nu}\frac{\partial^2\Theta}{\partial(y^+)^2}. \quad (9)$$

the constant $a/\nu = 1/Pr$ can be absorbed into the equation by putting

$$\sigma = y^+ Pr^{1/3} \tag{10}$$

and the function v_*/ν can be absorbed by putting

$$x^+ = \int_{x_0}^{x} \frac{v_*}{\nu}\,\mathrm{d}x. \tag{11}$$

With these substitutions, equation (9) becomes

$$\sigma \frac{\partial \Theta}{\partial x^+} = \frac{\partial^2 \Theta}{\partial \sigma^2}. \tag{12}$$

In order to take advantage of the observation that the problem must lead to a set of self-similar profiles, it is now necessary to find a combination of the independent variables $\eta(\sigma, x^+)$ which will reduce the second and third boundary condition (7a) to one. It is easy to see that any expression of the form

$$\eta = \frac{\sigma^m}{x^+} \tag{13}$$

will achieve this, since

$x = x_0$ corresponds to $x^+ = 0$ and renders $\eta = \infty$

$y = \infty$ corresponds to $\sigma = \infty$ and renders $\eta = \infty$.

The only remaining problem is to determine a value of the exponent m in equation (13) which will result in the transformation of equation (12) into an ordinary differential equation for $\Theta(\eta)$. By substitution, we find that

$$\eta = \frac{\sigma^3}{9x^+} = \frac{y^3 v_*^3 Pr}{9\nu^3 \int_{x_0}^{x} \frac{v_*\,\mathrm{d}x}{\nu}} \tag{14}$$

achieves our purpose, the numerical factor 9 having been added on aesthetic grounds. Introducing (8) we can also write

$$\eta = \frac{(y^+)^3 Pr}{9x^+}. \tag{14a}$$

Substitution of the appropriate form (14) into equation (12) or directly into equation (7b) leads to the ordinary differential equation

$$\eta \frac{\mathrm{d}^2\Theta}{\mathrm{d}\eta^2} + (\eta + \tfrac{2}{3}) \frac{\mathrm{d}\Theta}{\mathrm{d}\eta} = 0 \tag{15}$$

for the function $\Theta(\eta)$, with the boundary conditions

$$\left.\begin{aligned} \Theta &= 1 \text{ at } \eta = \infty \\ \Theta &= 0 \text{ at } \eta = 0 \end{aligned}\right\}.$$

This equation can be solved by repeated integration. Its first integral is

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\eta} = C_1 \eta^{-2/3} \exp(-\eta) \tag{16}$$

and the second integral is

$$\Theta(\eta) = C_2 + C_1 \gamma(\tfrac{1}{3}, \eta) \tag{17}$$

where

$$\gamma(a, \eta) = \textstyle\int_0^\eta \mathrm{e}^{-\lambda} \lambda^{a-1}\,\mathrm{d}\lambda \tag{17a}$$

is the incomplete gamma function [5, 6, 7]. Noting that

$$\gamma(a, \infty) = \Gamma(a) \text{ and that } \gamma(a, 0) = 0$$

it is easy to show that

$$\Theta(\eta) = \frac{\gamma(\frac{1}{3}, \eta)}{\Gamma(\frac{1}{3})} \tag{18}$$

constitutes the solution to our problem. The resulting universal temperature profile is shown plotted in Fig. 2. The values of the incomplete gamma function have been taken from [6] and the constant

$$\Gamma(\tfrac{1}{3}) = 2{\cdot}6784.$$

A short table for the function $\Theta(\eta)$ is also given, Table 1.

The rate of heat transfer is calculated from the heat flux $\dot{q}_w$ at the wall,

$$\dot{q}_w = k \left(\frac{\partial \theta}{\partial y}\right)_{y=0} = k\theta_\infty \left(\frac{\mathrm{d}\Theta}{\mathrm{d}\eta} \cdot \frac{\partial \eta}{\partial y}\right)_{y=0}. \tag{19}$$

It is noted that at $y = 0$, $\mathrm{d}\Theta/\mathrm{d}\eta$ is singular, but $\partial\theta/\partial y$ is regular. Referring to equation (16), we can write

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\eta} = \frac{\exp(-\eta)}{\eta^{2/3}\Gamma(\frac{1}{3})} \tag{19a}$$

and from equation (14)

$$\frac{\partial \eta}{\partial y} = \frac{y^2 v_*^3 Pr}{3\nu^3 x^+}. \tag{19b}$$

Hence

$$\frac{\partial \Theta}{\partial y} = \frac{3^{1/3} v_* Pr^{1/3} \exp(-\eta)}{\Gamma(\frac{1}{3}) \cdot \nu \cdot (x^+)^{1/3}} \tag{19c}$$

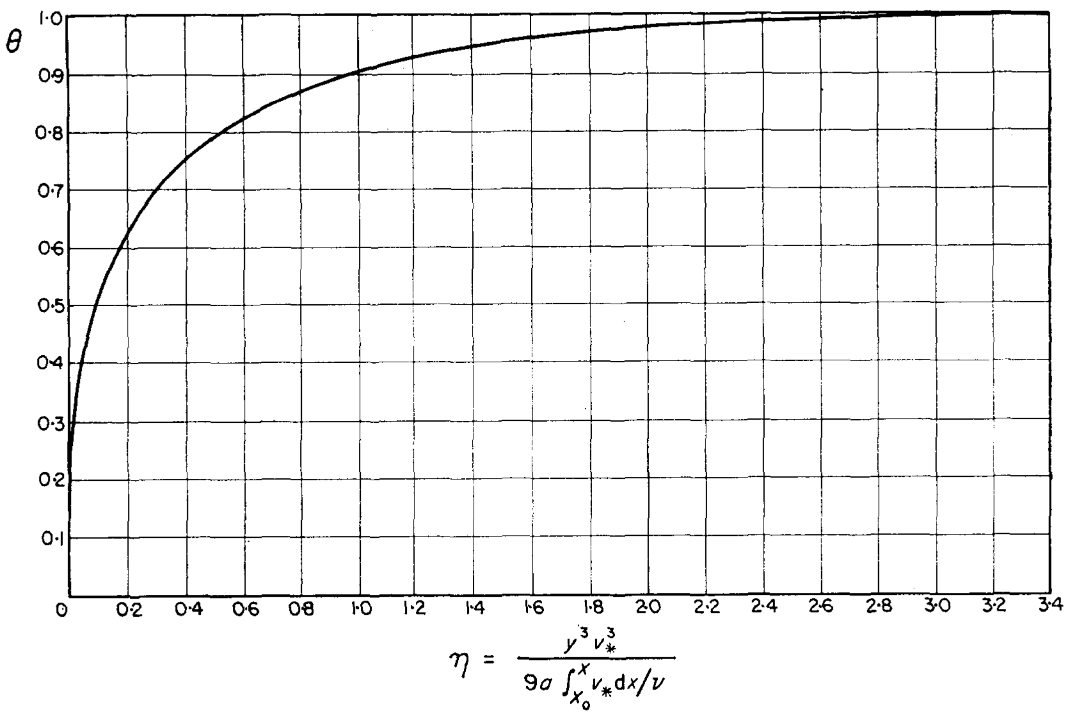


FIG. 2. The universal temperature profile.

Table 1. Values of the function $\Theta(\eta)$, equation (18)

η	$\Theta(\eta)$
0·00	0·0000
0·02	0·3025
0·04	0·3793
0·06	0·4320
0·08	0·4732
0·10	0·5073
0·12	0·5364
0·14	0·5620
0·16	0·5848
0·18	0·6054
0·20	0·6240
0·40	0·7496
0·60	0·8203
0·80	0·8694
1·00	0·9032
1·20	0·9272
1·40	0·9448
1·60	0·9576
1·80	0·9670
2·00	0·9743
2·50	0·9862
3·00	0·9924
3·50	0·9958
4·00	0·9976

and finally, from equation (19) it is seen that

$$\dot{q}_w = \frac{3^{1/3} v_*/\nu}{\Gamma(\frac{1}{3})} \cdot \frac{k\theta_\infty Pr^{1/3}}{(x^+)^{1/3}}. \qquad (20)$$

The most convenient dimensionless form is obtained by introducing the local Stanton number,

$$St_x = \frac{\dot{q}_w}{\rho c_p \bar{U}_\infty \theta_\infty} = 0{\cdot}53835 \frac{Pr^{-2/3}\sqrt{(\frac{1}{2}c_f)}}{(x^+)^{1/3}} \qquad (21)$$

where

$$3^{1/3}/\Gamma(\tfrac{1}{3}) = 0{\cdot}53835. \qquad (21a)$$

It is now a simple matter to derive expressions for the local Nusselt number Nu_x' which may be more convenient in applications. A short derivation yields the formula

$$Nu_x' = \frac{\dot{q}_w(x - x_0)}{k\theta_\infty}$$
$$= 0{\cdot}53835\,(\tfrac{1}{2}c_f)^{1/2} Re_x'(Pr/x^+)^{1/3}, \qquad (22)$$

where

$$Re_x' = \frac{\bar{U}_\infty(x - x_0)}{\nu}.$$

Average Nusselt or Stanton numbers cannot be computed at this stage explicitly, because their values depend on the variation of $v_*(x)$ along the flow. They will be discussed in connection with particular examples later, when more directly applicable equations will be given [section 3].

When computing temperature profiles, it is found that existing tables [6] are inadequate at small values of η. It is then useful to remember that [5, 7]

$$\gamma(1/3, \eta) = 3\eta^{1/3}(1 - \tfrac{1}{4}\eta + \tfrac{1}{14}\eta^2 + \dots) \quad (23)$$

3. SPECIALIZATION TO THE CASE OF A FLAT PLATE AT ZERO INCIDENCE

3.1. *Isothermal plate*

The preceding results will now be applied to a flat plate, and two cases will be considered. In the first case it will be assumed that the thermal boundary layer, $\delta_T(x)$, develops from the edge onwards, Fig. 3, so that $x_0 = 0$, and the plate is isothermal. The velocity boundary layer, $\delta(x)$, will start by being laminar, will undergo transition, and then become turbulent. This, however, has no bearing on the resulting solutions, except for affecting the form of $v_*(x)$, since all along the plate the thermal boundary layer will develop in regions where the velocity profile $u(y)$ is linear.* Since not enough is known about the conditions in the transition zone, it will be assumed that transition is sharp, occurs at a definite critical Reynolds number and that its value has been determined independently.

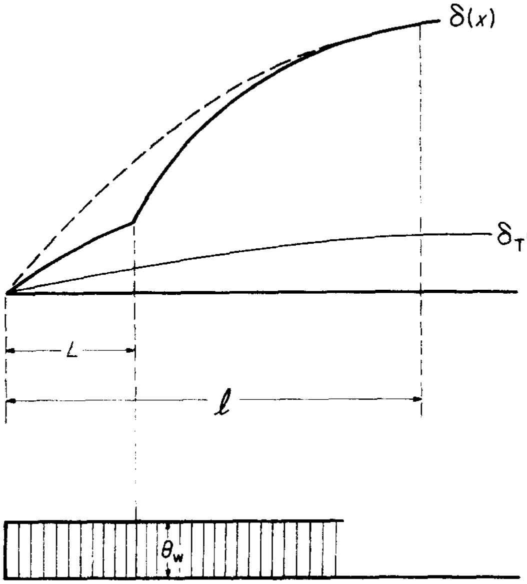


FIG. 3. Isothermal case.

In the laminar range

$$v_* = a^{1/2}\nu^{1/4}\bar{U}_\infty^{3/4}x^{-1/4}$$

with

$$a = 0{\cdot}332.$$

Then

$$x^+ = \int_0^x v_*(x)\,dx/\nu = \tfrac{4}{3}\,a^{1/2}\,\nu^{-3/4}\,\bar{U}_\infty^{3/4}\,x^{3/4}$$

and the local Stanton number is given by

$$St_x = 0{\cdot}339\,Pr^{-2/3}Re_x^{-1/2}. \quad (25)$$

where now

$$Re_x = \frac{\bar{U}_\infty x}{\nu}.$$

It is remarkable to note how little this expression differs from the Blasius solution for the flat plate [3],

$$St_x = 0{\cdot}332\,Pr^{-2/3}Re_x^{-1/2} \quad (25a)$$

which is not restricted to high Prandtl numbers. The form of the equation is reproduced accurately, and only the numerical constant is too high by about 2 per cent. Since equation (25a) can also be derived from the Reynolds analogy

$$St_x = \tfrac{1}{2}c_f$$

by adding the empirical factor $Pr^{-2/3}$ to it, equation (25) demonstrates that the resulting relation remains quite accurate for very high Prandtl numbers.

In the turbulent range, the shearing stress can be adequately approximated by the equation derived from the $\frac{1}{7}$-th power law, on condition that the Reynolds number is not too high. Thus

$$v_* = (0{\cdot}0296)^{1/2}\,\bar{U}_\infty^{9/10}\nu^{1/10}x^{-1/10}. \quad (26)$$

* The persistence of a linear segment of the velocity profile at the wall throughout the transition zone has been observed by Klebanoff who was kind enough to communicate this matter privately to one of us (J.K.).

It was shown by Prandtl [3] that a turbulent boundary layer on a flat plate behaves as if it had started at the leading edge*, consequently

$$x^- = \int_0^x v_*(x)\,\mathrm{d}x/\nu$$
$$= \tfrac{10}{9}(0{\cdot}0296)^{1/2}\,\nu^{-9/10}\,\bar{U}_\infty^{-9/10}\,x^{9/10}$$

and

$$St_x = 0{\cdot}161(Pr)^{-2/3}Re_x^{-2/5}. \qquad (27)$$

There is no difficulty in computing the average values, and in this case it is perhaps more convenient to employ the Nusselt number based on the total length l in the working formulae. For a laminar portion of length l, we obtain

$$Nu_l = 0{\cdot}679\,Pr^{1/3}\,Re_l^{1/2}.$$

For a plate of length l on which the laminar portion extends over a length $L(L < l)$, we obtain

$$Nu_l = \left\{\frac{0{\cdot}677}{Re_{\text{crit}}^{1/2}} + \frac{0{\cdot}268}{Re_l^{2/5}}\left[1 - \left(\frac{L}{l}\right)^{3/5}\right]\right\}Re_l\,.\,Pr^{1/3} \qquad (29)$$

where

$$Re_{\text{crit}} = \frac{\bar{U}_\infty L}{\nu}$$

is the Reynolds number for the point of transition, and the subscript l denotes that the respective groups refer to l as the characteristic length.

Finally, for a flat plate with a tripped boundary layer, i.e. a turbulent boundary layer starting at the leading edge

$$Nu_x = 0{\cdot}161\,Re_x^{3/5}\,Pr^{1/3} \qquad (30)$$

and

$$Nu_l = 0{\cdot}268\,Re_l^{3/5}\,Pr^{1/3}. \qquad (30a)$$

This result can be used to evaluate the applicability of the Prandtl–Taylor or von Kármán [3] theories of heat transfer on a flat plate. It is well-known that the above theories are applicable only to Prandtl number differing little from unity [8] when they can all be fitted numerically by the equation

$$Nu_x = 0{\cdot}0296\,Re_x^{0{\cdot}8}\,Pr^{1/3}. \qquad (30b)$$

A comparison of equations (30) and (30b) shows that the elementary theories reproduce correctly the dependence on the Prandtl number, but that the Reynolds number dependence is overestimated, the exponent being equal to 0·8 instead of 0·6; they cannot, therefore, be applied to fluids with high Prandtl numbers. The two relations, (30) and (30b) provide, therefore, an upper and a lower bound for the Nusselt number, when the Prandtl number lies in a range intermediate between unity and a very large value.

3.2. *Step in wall temperature*

The next case of importance occurs when the temperature along the plate is not uniform, but undergoes a step-like change from $T_w = T_\infty$ $(\Theta = 1)$ at $x \le x_0$ to $T_w \ne T_\infty$ $(\Theta = 0)$ at $x > x_0$. Depending on the position of x_0 with respect to the point of transition at $x = L$, it is necessary to distinguish a number of specific cases. Each case involves rather obvious integrations which need not be discussed in detail. The results of these integrations are given in Table 2 which includes the cases discussed in section 3.1 for the sake of completeness.

4. VARIABLE WALL TEMPERATURE

4.1. *Lighthill's solution*

Lighthill's original paper [1] contained an extension of the preceding theory to the case of an arbitrary temperature distribution along the wall. The local heat flux $\dot{q}_w$ at a position x along a cylindrical wall having an arbitrary temperature distribution $T_w(x)$ from section $x_0 = 0$ onwards was given in the form of the following Stieltjes integral

$$\dot{q}_w(x) = -k\,\frac{3^{1/3}Pr^{1/3}v_*(x)}{\Gamma(\frac{1}{3})\,.\,\nu}\int_0^x \frac{\mathrm{d}\vartheta(\xi)}{[x^+(x) - x^+(\xi)]^{1/3}}. \qquad (31)$$

* This assumption is sufficiently accurate for many purposes. In actual fact, the virtual beginning of the turbulent boundary layer is not exactly at the leading edge, and this circumstance can be easily allowed for if necessary.

Table 2. Summary of formulae for isothermal plates and for plates with a step-like change in wall temperature

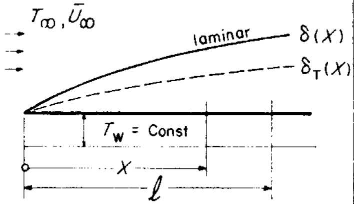
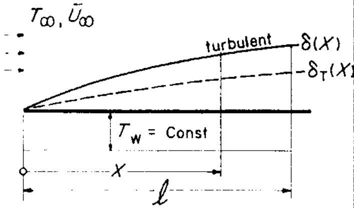
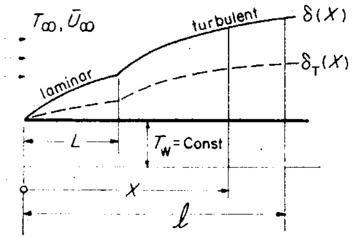
NO	CASE	LOCAL STANTON NUMBER St_x	AVERAGE NUSSELT NUMBER Nu_ℓ	DEFINITIONS
1		$0{\cdot}339\, Pr^{-\frac{2}{3}} Re_x^{-\frac{1}{2}}$	$0{\cdot}679\, Pr^{\frac{1}{3}} Re_\ell^{\frac{1}{2}}$	$Re = \frac{\bar{U}_\infty x}{\nu}$ $Nu_\ell = Re_\ell Pr \frac{1}{\ell} \int_0^\ell St\, dx$
2		$0{\cdot}161\, Pr^{-\frac{2}{3}} Re_x^{-\frac{2}{5}}$	$0{\cdot}268\, Pr^{\frac{1}{3}} Re_\ell^{\frac{3}{5}}$	
3		$x < L$ $0{\cdot}339\, Pr^{-\frac{2}{3}} Re_x^{-\frac{1}{2}}$ $x > L$ $0{\cdot}161\, Pr^{-\frac{2}{3}} Re_x^{-\frac{2}{5}}$	$x > L$ $Re_\ell Pr^{\frac{1}{3}} \Big\{0{\cdot}677\, Re_{crit}^{-\frac{1}{2}}$ $+ 0{\cdot}268\, Re_\ell^{-\frac{2}{5}} \Big[1 - \left(\frac{L}{\ell}\right)^{\frac{3}{5}}\Big]\Big\}$	$Re_{crit} = \frac{\bar{U}_\infty L}{\nu}$ $Re_\ell = \frac{\bar{U}_\infty \ell}{\nu}$

Table 2—continued

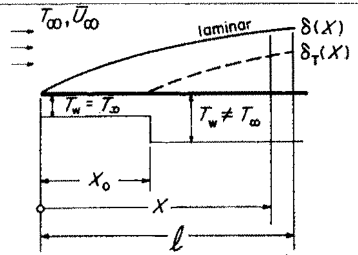
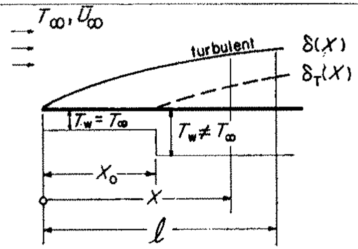
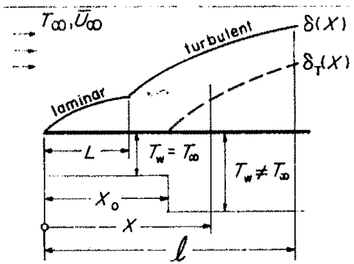
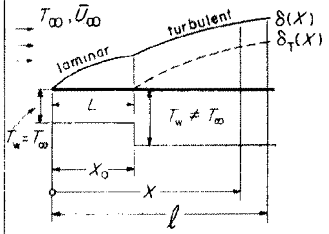
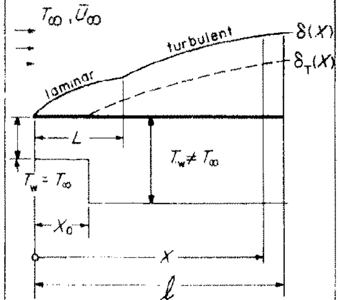
NO	CASE	LOCAL STANTON NUMBER St_x	AVERAGE NUSSELT NUMBER Nu_ℓ	DEFINITIONS
4		$0{\cdot}339 Pr^{-\frac{2}{3}} Re_x^{-\frac{1}{2}} \left[1 - \left(\frac{X_0}{X}\right)^{\frac{3}{4}}\right]^{-\frac{1}{3}}$	$0{\cdot}677 Pr^{\frac{1}{3}} Re_{x_0}^{\frac{1}{2}} \left[\left(\frac{\ell}{X_0}\right)^{\frac{3}{4}} - 1\right]^{\frac{2}{3}}$	$Re_{x_0} = \frac{\bar{U}_\infty x_0}{\nu}$
5		$0{\cdot}161 Pr^{-\frac{2}{3}} Re_x^{-\frac{2}{5}} \left[1 - \left(\frac{X_0}{X}\right)^{\frac{9}{10}}\right]^{-\frac{1}{3}}$	$0{\cdot}268 Pr^{\frac{1}{3}} Re_{x_0}^{\frac{3}{5}} \left[\left(\frac{\ell}{X_0}\right)^{\frac{9}{10}} - 1\right]^{\frac{2}{3}}$	
6				

Table 2—continued

NO	CASE	LOCAL STANTON NUMBER St_x	AVERAGE NUSSELT NUMBER Nu_{ℓ}	DEFINITIONS
7		$0{\cdot}161\,Pr^{-\frac{2}{3}} Re_x^{-\frac{2}{5}} \left[1-\left(\frac{L}{X}\right)^{\frac{9}{10}}\right]^{-\frac{1}{3}}$	$0{\cdot}268\,Pr^{\frac{1}{3}} Re_{crit}^{\frac{3}{5}} \left[\left(\frac{\ell}{L}\right)^{\frac{9}{10}}-1\right]^{\frac{2}{3}}$	
8		$L < X < \ell\ ;\ \ell > L$ $0{\cdot}161\,Pr^{-\frac{2}{3}} Re_x^{-\frac{2}{5}} \left[1-\left(\frac{X_0}{X}\right)^{\frac{9}{10}}\right]^{-\frac{1}{3}}$	$Pr^{\frac{1}{3}} \Big\{0{\cdot}677\,Re_{X_0}^{\frac{1}{2}} \left[\left(\frac{L}{X_0}\right)^{\frac{3}{4}}-1\right]^{\frac{2}{3}}$ $+\,0{\cdot}268\,Re_{X_0}^{\frac{3}{5}} \Big[\Big\{\left(\frac{\ell}{X_0}\right)^{\frac{9}{10}}-1\Big\}^{\frac{2}{3}}$ $-\Big\{\left(\frac{L}{X_0}\right)^{\frac{9}{10}}-1\Big\}^{\frac{2}{3}}\Big]\Big\}$	When $X_0 < X < L$ and $X_0 < \ell < L$ see CASE 4

Lighthill's equation (29) of Ref. 1 has been transcribed here in our notation in order to clarify the sequence of operations required to evaluate $\dot{q}_w(x)$. First, Fig. 4, it should be noted that here

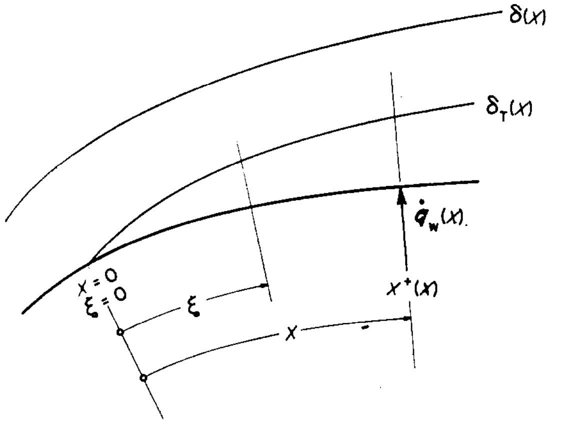


FIG. 4. Notation for Lighthill's solution, equation (31).

$$\vartheta = T - T_\infty \tag{32}$$

and that, in general, in problems involving a variable wall temperature $T_w(x)$, the definition (32) is more convenient than that in equation (2) because only cases in which $T_\infty = \text{const}$ are of practical interest. We shall give preference to the variable ϑ over θ in the present section. Secondly, it should be noted that $x^+(x)$, defined in equation (11) denotes a constant value in the process of integration, namely the value of x^+ at the station at which $\dot{q}_w$ is being evaluated. The quantity $x^+(\xi)$ is defined as

$$x^+(\xi) = \int_0^\xi \frac{v_*(\xi)}{\nu}\,\mathrm{d}\xi$$

and denotes a function, ξ being the dummy variable of integration in equation (31).

The procedure to be adopted in evaluating the integral (31) depends on whether the temperature distribution $\vartheta(\xi)$ is continuous or whether it involves step-like jumps. In the former case $\mathrm{d}\vartheta(\xi) = (\xi)\,\mathrm{d}\xi$, and the evaluation is straightforward, except that one reservation must be made. Often, the function, $\vartheta(\xi)$ will be given by a discrete set of measured values, and the determination of $\vartheta(\xi)$ will require fairing. As a result, the calculation may be grossly in error. When the temperature distribution $\vartheta(\xi)$ involves discontinuities, in particular when it is given as a sequence of steps from measurements, it is necessary to perform the integration without introducing (x). This is best done by eliminating, for example graphically, the coordinate ξ from the two functions $x^+(\xi)$ and $\vartheta(\xi)$ and by integrating the resulting relation $[x^+(x) - x^+(\vartheta)]^{-1/3}$, in which $x^+(x)$ is a constant parameter as far as the integration is concerned, as already mentioned, and by integrating with respect to ϑ directly.

It is clear that either method is cumbersome in practice. The principal motivation for undertaking the calculations described in the remainder of this section was to arrive at formulae which could be more easily adapted to actual computations.

4.2. *Two steps in temperature*

The simplest practical case of variable wall temperature, apart from the ones considered in sections 2 and 3, involves a portion of the wall of length b heated to a constant temperature T_w different from the remainder which is maintained at T_∞, Fig. 5(a). The present case is also important in that it will illustrate the method of dealing with completely arbitrary wall-temperature distributions to be discussed in section 4.3. The general method makes systematic use of the fact that the energy equation (1) is linear and that superposition can be employed to generate complex solutions from the simple solution given in equation (18) of section 2.

It is now more convenient to put

$$\Theta = \frac{T - T_\infty}{T_w - T_\infty} = 1 - \frac{T_w - T}{T_w - T_\infty}$$

instead of the definition in equation (7). Equation (12) still remains valid in either of the two regions marked I and II in Fig. 5. However, owing to the change in the definition of Θ, the boundary conditions (15a) will change to

$$\left.\begin{aligned} \Theta &= 0 \text{ at } \eta = \infty \\ \Theta &= 1 \text{ at } \eta = 0 \end{aligned}\right\} \tag{34}$$

it being necessary to exercise some care in the interpretation of η for either of the two regions. The change in the boundary conditions has the

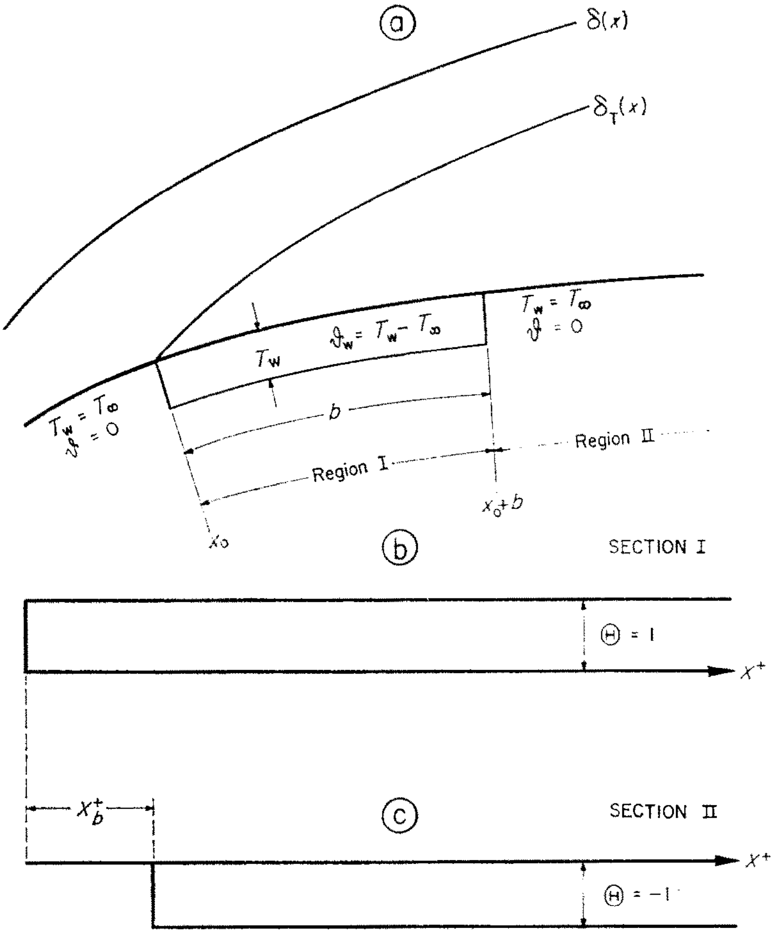


FIG. 5. Two steps in temperature.

effect of changing the solution in equation (18) to

$$\Theta(\eta) = \frac{\Gamma(\frac{1}{3}, \eta)}{\Gamma(\frac{1}{3})} \tag{35}$$

where

$$\Gamma(a, \eta) = \int_\eta^\infty e^{-\lambda} \int \lambda^{a-1}\, d\lambda = \Gamma(a) - \gamma(a, \eta) \tag{36}$$

is the complementary incomplete gamma function [7], and $\Gamma(a)$ denotes the ordinary gamma function.*

* This unfortunate notation, though accepted, may be somewhat confusing, but in this paper the two functions Γ will always be written with the parameters and variables shown in the parentheses to avoid misunderstandings.

The solution in region I remains unaffected by the second step, and is the same as for the case illustrated in Fig. 5(b). It is thus

$$\Theta_1 = \frac{1}{\Gamma(\frac{1}{3})} \Gamma\left(\tfrac{1}{3}, \frac{(y^+)^3 Pr}{9x^+}\right). \tag{37}$$

In region II it is necessary to take into account the second step in temperature which occurs at $x = x_0 + b$ or at $x^+ = x_b^+$. We therefore consider the case illustrated in Fig. 5(c) for which the solution will be identical with (37) except for the fact that the origin must be shifted to $x^+ = x_b^+$ so that x^+ must be replaced by $x^+ - x_b^+$ in the argument of the function, where

$$x_b^+ = \int_{x_0}^{x_0+b} \frac{v_*}{\nu}\, dx. \tag{38}$$

Furthermore, a negative sign must be appended since Θ decreases by unity instead of increasing. Superimposing the cases shown in Figs. 5(b) and 5(c), we obtain the boundary conditions which are valid in region II, Fig. 5(a). At $x = x_0 + b$, the temperature profile consists of Θ_1 with the boundary condition for case 5(c) namely $\Theta = 0$ superimposed on it, and is the one actually prevailing. Consequently, in region II, we have

$$\Theta_{\text{II}} = \frac{1}{\Gamma(\frac{1}{3})}\left\{\Gamma\left(\frac{1}{3}, \frac{(y^+)^3 Pr}{9x^+}\right) - \Gamma\left(\frac{1}{3}, \frac{(y^+)^3 Pr}{9(x^+ - x_b^+)}\right)\right\}. \tag{39}$$

The parameter x_b^+ defined in equation (38) and appearing in equation (39) characterizes the extent of the zone of elevated temperature along the wall. It is easy to see from equations (37) and (39) that the temperature field is continuous at $x = x_0 + b$, i.e. at the point where the wall temperature drops suddenly from T_w to T_∞, because

$$\Gamma(\tfrac{1}{3}, \infty) = 0.$$

Furthermore

$$\frac{\partial \Theta_{\text{I}}}{\partial x^+} = \frac{\partial \Theta_{\text{II}}}{\partial x^+} \text{ at } x = x_0 + b$$

at any value of y^+, because

$$\frac{\partial}{\partial \eta}\{\Gamma(a, \eta)\} = -\, e^{-\eta}\, \eta^{a-1} \to 0 \text{ as } \eta \to \infty.$$

The rate of heat transfer in region I is given by the formulae of section 2, whereas the rate of heat transfer in region II can be easily calculated by differentiation from equation (39). In this manner it is found that

$$\left(\frac{\partial \Theta_{\text{II}}}{\partial y^+}\right)_{y^+=0} = \frac{Pr^{1/3}(3)^{1/3}}{\Gamma(\frac{1}{3})(x^+)^{1/3}}\left[1 - \left(\frac{x^+}{x^+ - x_b^+}\right)^{1/3}\right]$$

$$= 0{\cdot}53835\,\frac{Pr^{1/3}}{(x^+)^{1/3}}\left[1 - \left(\frac{x^+}{x^+ - x_b^+}\right)^{1/3}\right], \tag{40}$$

(see equation (21a)), and that

$$\left(\frac{\partial \vartheta_{\text{II}}}{\partial y}\right)_{y=0} = -\,0{\cdot}53835\,\frac{v_* Pr^{1/3}}{\nu\,.\,(x^+)^{1/3}} \times \left[1 - \left(\frac{x^+}{x^+ - x_b^+}\right)^{1/3}\right].\,(T_w - T_\infty), \tag{41}$$

or

$$St_x = 0{\cdot}53835\,\frac{Pr^{-2/3}\sqrt{(\frac{1}{2}c_f)}}{(x^+)^{1/3}}\left[1 - \left(\frac{x^+}{x^+ - x_b^+}\right)^{1/3}\right]. \tag{42}$$

Since the problem under consideration involves a constant imposed temperature difference, $T_w - T_\infty$, the local rate of heat transfer $\dot{q}_w$, is proportional to it. Thus a Stanton number can be formed. In more complex cases the Stanton and Nusselt numbers lose their utility because such problems do not contain a physically meaningful characteristic temperature difference for reference.

It is interesting to examine the trend in the form of the temperature profiles downstream from the second step in temperature. This is best done by considering a specific example, Fig. 6,

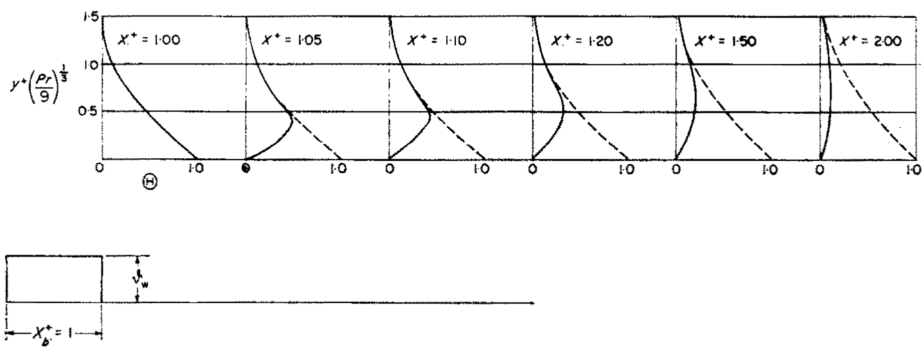


FIG. 6. Temperature profiles downstream of the second step.

with $x_b^+ = 1$. The temperature profiles have been plotted as $\Theta = (T - T_\infty)/(T_w - T_\infty)$ against $y^+ Pr^{1/3}/9^{1/3}$ for convenience, the plots including the values $x^+ = 1{\cdot}0$, $1{\cdot}1$, $1{\cdot}2$, $1{\cdot}5$ and $2{\cdot}0$. It is interesting to note how fast the disturbance caused by the second step disappears on progression in the downstream direction. The broken curves in Fig. 6 represent temperature profiles which would exist at the respective positions x^+ if the wall temperature ϑ_w continued at its constant value without suddenly decreasing to zero at $x^+ = 1$. A comparison of the two sets of profiles gives an idea of how deeply the sudden drop in temperature penetrates into the boundary layer.

4.3. *Arbitrary temperature distribution; step-wise approximation*

The preceding case can be easily generalized to an arbitrary temperature distribution, Fig. 7, if the latter is replaced by a sequence of discrete jumps. The partial solution for the n-th jump at

$$x^+ = x_{b,n}^- = \int_{x_0}^{(x_0 + b_n)} \frac{v_*}{\nu}\,\mathrm{d}x$$

can be written

$$\vartheta_n = \frac{1}{\Gamma(\frac{1}{3})}\,\Gamma\left(\tfrac{1}{3}, \frac{(y^{+3})Pr}{9(x^+ - x_{b,n}^+)}\right) . \Delta\vartheta_n. \quad (43)$$

The complete solution for the interval

$$x_{b,n}^- < x^+ < x_{b,(n+1)}^+ \quad (43a)$$

is thus seen to be

$$\vartheta = \frac{1}{\Gamma(\frac{1}{3})} \sum_{n=1}^{n} \left\{ \Gamma\left(\tfrac{1}{3}, \frac{(y^+)^3 Pr}{9(x^+ - x_{b,n}^+)}\right) . \Delta\vartheta_n \right\}. \quad (44)$$

It is equally easy to write down the expression for the local heat flux, namely

$$\dot{q}_w(x) = -k\left(\frac{\partial\vartheta}{\partial y}\right)_{y=0}$$

$$= -0{\cdot}53835\,\frac{k v_*(x) . Pr^{1/3}}{\nu} \sum_{n=1}^{n} \frac{\Delta\vartheta_n}{(x^+ - x_{b,n}^+)^{1/3}} \quad (45)$$

in the interval (43a). Alternatively

$$\dot{q}_w(x) = -0{\cdot}53835\,\rho c_p v_*(x) Pr^{-2/3} \times \sum_{n=1}^{n} \frac{\Delta\vartheta_n}{(x^+ - x_{b,n}^+)^{1/3}}. \quad (45a)$$

It is easy to verify that equation (45a) reduces to equation (42) for the case of two consecutive equal steps of opposite sign. There is little

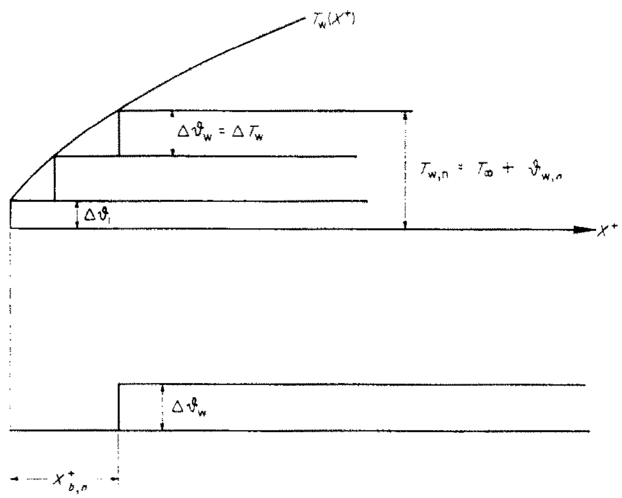


FIG. 7. Arbitrary temperature distribution.

advantage in representing the heat flux in the form of a Stanton number, because now $\Delta\vartheta_n$ denotes an arbitrarily chosen step in temperature to which no physical meaning can be attached, as anticipated in section 4.2, and because the local rate of heat transfer does not appear in a form involving the local temperature difference $\vartheta_{w,n}$.

The preceding formulae are easy to apply in practice, and for a given smooth temperature variation, it is always possible to inscribe as well as to circumscribe a step polygon, Fig. 8, in order to obtain two bounds for $\dot{q}_w(x)$. The total rate of heat transfer $\dot{Q}_w$ is obtained by integration in the usual way.

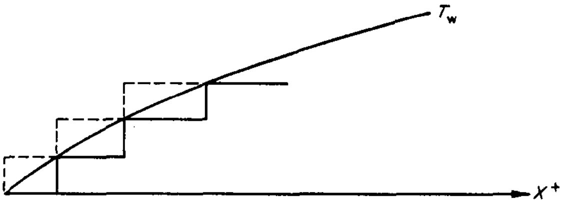


FIG. 8. Two bounds for the solution.

The relation in equation (45a) provides a complete solution to our problem, but there is some advantage in adopting a more analytic approach which leads to an alternative form of the solution.

4.4. *Arbitrary temperature distribution; analytic solution*

When the temperature distribution along the wall, $\vartheta_w(x)$, is given in an analytic form, it might appear that the most convenient method of writing the solution is simply to pass to the limit of $\Delta\vartheta_n \to 0$ in equation (44) or (45a) thus replacing the sums by integrals. A straightforward passage to the limit reduces equation (45a) to Lighthill's form, equation (31), and this would cause the difficulties discussed in section 4.1. It is, therefore, preferable to start with equation (12), which is seen to be valid for ϑ, i.e. with

$$\sigma \frac{\partial\vartheta}{\partial x^+} = \frac{\partial^2\vartheta}{\partial\sigma^2}. \tag{46}$$

This being a linear equation, it is natural to solve it by the use of a transform, and in this connection the Hankel transform proves to be very convenient. It would be tedious to repeat here the otherwise standard computations, and it is sufficient to quote the final result, namely

$$\vartheta(\sigma, x^+) = \frac{1}{\Gamma(\frac{1}{3})} \int_{\sigma^3/9x^+}^{\infty} \left[\vartheta_w\left(x^+ - \frac{\sigma^3}{9t}\right)\right] \times e^{-t}\, t^{-2/3}\, dt, \tag{47}$$

or alternatively

$$\vartheta(\sigma, x^+) = \frac{\sigma}{3^{8/3}\Gamma(\frac{1}{3})} \int_0^{x^+} \frac{\vartheta_w(z) \exp\,[-\sigma^3/9(x^+ - z)]\, dz}{(x^+ - z)^{4/3}}, \tag{47a}$$

where $\vartheta_w(z)$ is the variation of the wall temperature ϑ_w expressed in terms of the variable x^+, i.e. $\vartheta_w(x^+)$.

It can be verified that either of the forms (47) or (47a) satisfies the differential equation (46) and the boundary conditions

$$\vartheta = 0 \text{ for } x^+ \to 0 \text{ and } \sigma \to \infty, \text{ i.e. } y^+ \to \infty$$
$$\vartheta = \vartheta_w(x^+) \text{ for } \sigma \to 0 \text{ and any } x^+, \text{ i.e. for } y^+ \to 0. \tag{47b}$$

Substituting σ from equation (10) and noting that

$$\frac{1}{3^{8/3}\Gamma(\frac{1}{3})} = 0{\cdot}019928 \tag{47c}$$

we can also write

$$\vartheta(x^+, y^+) = 0{\cdot}019928\, y^+ Pr^{1/3} \times \int_0^{x^+} \frac{\vartheta_w(z) \exp\,[-(y^+)^3 Pr/9(x^+ - z)]\, dz}{(x^+ - z)^{4/3}}.$$

The gradient of temperature at the wall involves the factor

$$\frac{\partial\vartheta}{\partial\sigma} = -\frac{3^{1/3}}{\Gamma(1/3)} \frac{\vartheta_w(0) \exp\,(-\sigma^3/9x^+)}{(x^+)^{1/3}} + \int_0^{x^+} \frac{\vartheta_w(z)\,.\,\exp\,[-\sigma^3/9(x^+ - z_0)]\, dz}{(x^+ - z)^{1/3}} \tag{48}$$

in which $\sigma \to 0$ must be substituted. The expression has been established for the case when $\vartheta_w(x)$ undergoes a jump $\vartheta_w(0)$ at $x^+ = 0$ and then varies continuously in a manner prescribed by $\vartheta_w(x)$. The local heat flux is then

$$\dot{q}_w(x) = 0{\cdot}53835\ \rho c_p v_*(x) Pr^{-2/3} \times \left[\frac{\vartheta_w(0)}{(x^+)^{1/3}} + \int_0^{x^+(x)} \frac{\vartheta_w(z)\,\mathrm{d}z}{(x^+ - z)^{1/3}}\right]. \qquad (49)$$

Readers familiar with the theory of Stieltjes integrals might have been in a position to write down equation (49) directly from equation (31) derived by Lighthill, realizing that the first term in square brackets is the contribution due to the initial jump $\vartheta_w(0)$, and that the second term in it, the integral, represents the contribution of the continuous part of the temperature variation, all upstream of the position x. It is seen from equation (49) that the local heat flux is no longer proportional even to the local temperature difference $\vartheta_w(x)$, but depends in an intimate way on the whole of the upstream temperature distribution $\vartheta_w(x)$.

5. SPECIAL CASES OF PRACTICAL IMPORTANCE

Considering the large number of possible temperature distributions as well as the possible forms of $v_*(x)$, it is realized that the formulae of the preceding section cannot be specialized to the same extent as those in section 3. Nevertheless, in order to facilitate practical computations, it is useful to record explicit expressions for wall temperature distributions given by a power series. Considering one term of the series, i.e.

$$\vartheta_w = A_n (x^+)^n,$$

where n is an integer, it is possible to show directly by substitution into equation (49) that

$$\dot{q}_w(x) = 0{\cdot}53835\ \rho c_p v_*(x) Pr^{-2/3} A_n \times \frac{\Gamma(n+1)\Gamma(\frac{2}{3})}{\Gamma(n+\frac{2}{3})} \cdot \frac{(x^+)^n}{(x^+)^{1/3}} = 0{\cdot}72901\ \frac{\Gamma(n+1)}{\Gamma(n+\frac{2}{3})} \times \rho c_p v_*(x) Pr^{-2/3} \frac{\vartheta_w(x)}{(x^+)^{1/3}}. \qquad (50)$$

For the power series

$$\vartheta_w = \sum_{n=0}^{n} A_n (x^+)^n$$

we obtain the sum

$$\dot{q}_w(x) = 0{\cdot}72901\ \frac{\rho c_p v_*(x) Pr^{-2/3}}{(x^+)^{1/3}} \times \sum_{n=0}^{n} \frac{\Gamma(n+1)}{\Gamma(n+\frac{2}{3})} A_n \,.\, (x^+)^n. \qquad (51)$$

Finally, for

$$\vartheta_w = Ax^+ - B,$$

we have

$$\dot{q}_w(x) = 0{\cdot}53835\ \rho c_p v_*(x) \frac{Pr^{-2/3}}{(x^+)^{1/3}} (\tfrac{3}{2} Ax^+ - B). \qquad (52)$$

Some readers may prefer to approximate an empirical temperature distribution $\vartheta_w(x)$, expressed in the form $\vartheta_w(x^+)$, by straightline segments rather than by steps. In such cases equation (51) will prove to be of value.

ACKNOWLEDGEMENT

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Résumé—L'article considère un problème qui a été traité mathématiquement pour la première fois par Lighthill pour des conditions physiques différentes. Des solutions sont données dans le cas de la convection forcée à travers une couche limite turbulente quand $Pr \to \infty$, c'est-à-dire quand la couche limite thermique est entièrement à l'intérieur d'une sous-couche laminaire dans laquelle le profil des vitesses est linéaire.

Le cas d'une plaque plane à température uniforme ou avec un échelon de température est traité très en détail et une tabulation commode de la formule est donnée pour un certain nombre de cas.

Le cas d'une température de paroi variable est résolu de deux façons. 1—la distribution de température est remplacée par une suite d'échelons que l'on superpose; 2—une solution analytique exacte est donnée dans le cas où la fonction température consiste en un échelon suivi par une distribution donnée analytiquement. Dans le dernier cas, les équations sont données sous forme analytique pour une variation de température représentée par un polynôme dont un cas particulier est la variation de temperature linéaire.

Zusammenfassung—Es wird ein Problem behandelt, das zuerst mathematisch von Lighthill in anderem physikalischen Zusammenhang gebracht worden war. Lösungen sind für den Sonderfall der Zwangskonvektion bei turbulenter Grenzschicht und $Pr \to \infty$ angegeben, d.h. für den Fall, dass die thermische Grenzschicht vollständig auf die laminare Unterschicht von linearem Geschwindigkeitsprofil beschränkt bleibt.

Die ebene Platte mit gleichmässiger Temperaturverteilung oder mit einem Temperatursprung wird ausführlich behandelt und eine übersichtliche Formelzusammenstellung für eine Reihe von Fällen angegeben.

Der Fall der veränderlichen Wandtemperatur wird auf zweifache Weise gelöst. Erstens, indem die Temperaturverteilung durch eine Schrittfolge mit nachfolgender Superposition ersetzt wird. Zweitens, indem eine exakte analytische Lösung angegeben werden kann, wenn die Temperaturfunktion aus einem Sprung besteht, dem eine gegebene analytische Verteilung folgt. Für letzteren Fall ist eine Gleichung für eine polynome Temperaturänderung in geschlossener Form gegeben, Die lineare Temperaturänderung ist davon ein spezieller Fall.

Аннотация—В статье рассматривается задача, математическое решение которой было получено Лайтхиллом при другом физическом содержании. Решения даны для предельного случая вынужденной конвекции через турбулентный пограничный слой при $Pr \to \infty$, т.е. когда тепловой пограничный слой полностью содержится в ламинарном подслое с линейным профилем скорости.

Даётся детальное рассмотрение случая плоской пдастины при постоянной и ступенчатой изменяющейся температуре, а для ряда случаев формулы сведены в таблицы, удобные для применения.

Случай переменной температуры стенки решается двумя способами. Во-первых, распределение температур заменяется последовательностью ступеней и используется их суперпозиция. Во-вторых, даётся точное аналитическое решение для случая, когда температурная функция состоит из ступеньки, за которой следует данное аналитическое распределение. В последнем случае дана замкнутая форма уравнений, когда температура изменяется как полином. Частным случаем этого изменения является линейное изменение температуры.